

# Weak Closure Theorem for Double Staircase Actions

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## 1 Introduction

Considering measure-preserving transformations of a Probability space  $(X, \mu)$  we introduce a double staircase construction  $T$  and show that its semi-group of all weak limits of powers ( $WLP(T)$ ) is

$$\{\Theta, 2^{-m}T^n + (1 - 2^{-m})\Theta : m \in \mathbf{N}, n \in \mathbf{Z}\},$$

where  $\Theta$  stands for the orthogonal projection  $L_2(X, \mu)$  onto the space of constant functions. Mixing sequences are controlled via Adams' method [1], non-mixing ones come to light by means of secondary limit methods (see [4]).

**Staircase rank one transformation** is determined by an integer  $h_1$  and a sequence  $r_j$  of cuttings. We recall its definition. Let our  $T$  on the step  $j$  be associated with a collection of disjoint intervals

$$E_j, TE_j, T^2E_j, \dots, T^{h_j}E_j.$$

We cut  $E_j$  into  $r_j$  subintervals of the same measure

$$E_j = E_j^1 \sqcup E_j^2 \sqcup E_j^3 \sqcup \dots \sqcup E_j^{r_j},$$

then for all  $i = 1, 2, \dots, r_j$  we consider columns

$$E_j^i, TE_j^i, T^2E_j^i, \dots, T^{h_j}E_j^i.$$

Adding over  $i$ -th column  $i - 1$  spacers we obtain a partition

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$$\begin{aligned}
& E_j^1, TE_j^1, T^2 E_j^1, \dots, T^{h_j-1} E_j^1, T^{h_j} E_j^1, \\
& E_j^2, TE_j^2, T^2 E_j^2, \dots, T^{h_j-1} E_j^2, T^{h_j} E_j^2, T^{h_j+1} E_j^2, \\
& E_j^3, TE_j^3, T^2 E_j^3, \dots, T^{h_j-1} E_j^3, T^{h_j} E_j^3, T^{h_j+1} E_j^3, T^{h_j+2} E_j^3, \\
& \dots \quad \dots \quad \dots \\
& E_j^r, TE_j^r, T^2 E_j^r, \dots, T^{h_j-1} E_j^r, T^{h_j} E_j^r, T^{h_j+1} E_j^r, T^{h_j+2} E_j^r, \dots, T^{h_j+r-1} E_j^r,
\end{aligned}$$

where  $r = r_j$ . For all  $i < r_j$  we set  $T^{h_j+i} E_j^i = E_j^{i+1}$ .

Thus, we get  $j+1$ -the tower  $E_{j+1}, TE_{j+1}, T^2 E_{j+1}, \dots, T^{h_{j+1}} E_{j+1}$  with

$$E_{j+1} = E_j^1, \quad h_{j+1} + 1 = (h_j + 1)r_j + \sum_{i=1}^{r_j-1} i.$$

This staircase construction is a special case ( $s(i) = i - 1$ ) of a general rank one construction with a sequence  $\bar{s}_j$  of spacer vectors

$$\bar{s}_j = (s_j(1), s_j(2), \dots, s_j(r_j - 1), s_j(r_j)).$$

**Double staircase transformation** is defined by an integer  $h_1$  and a spacer sequence

$$\bar{s}_j = (0, 1, 2, 3, \dots, r'_j - 2, r'_j - 1, 0, 1, 2, 3, \dots, r'_j - 2, r'_j - 1),$$

where  $2r'_j = r_j$ . We presume that  $r_j \rightarrow \infty$ . In what follows we presume that Adams' condition  $r_j^2/h_j \rightarrow 0$  is satisfied. This restriction plays only a technical role and will be used implicitly in approximation formulas.

**On notations.** We write  $a(j) \approx b(j)$  instead of  $a(j) - b(j) \rightarrow 0$  (or  $\frac{a(j)}{b(j)} \rightarrow 1$ ) and use weak  $\approx_w$  and strong  $\approx$  operator approximations.

**Main result.**

**THEOREM 1.** *A double staircase transformation  $T$  possesses the following semi-group of weak limits of its powers (WLP):*

$$\{\Theta, 2^{-m}T^n + (1 - 2^{-m})\Theta : m \in \mathbf{N}, n \in \mathbf{Z}\},$$

where  $\Theta$  is the orthogonal projection into the space of constant functions.

**LEMMA 1.**  *$M = \mathbf{N} \setminus \cup_j [0.5h_{j+1} - h_j, 0.5h_{j+1} + h_j]$  is a mixing set: if  $m_i \rightarrow \infty$  and  $m_i \in M$ , then  $T^{m_i} \rightarrow \Theta$ .*

Thus, if  $n_j$  is a non-mixing sequence,  $|n_j| \in [h_{j'}, h_{j'+1})$ , then

$$|n_j| \in [0.5h_{j'+1} - h_{j'}, 0.5h_{j'+1} + h_{j'}].$$

We prove Lemma 1 in section 3.

## 2 Non-mixing sequences

An operator of multiplication by  $\chi_D$  is denoted by  $\hat{D}$ .

**LEMMA 2.** *Let  $\mu(D(j)) \rightarrow a > 0$  and  $\mu(D(j)\Delta TD(j)) \rightarrow 0$ . Then  $\hat{D}(j) \approx_w aI$ . If  $Q(j) \approx \Theta$ , then  $\hat{D}(j)Q(j) \approx_w a\Theta$ .*

A proof of this lemma is an exercise.

**Example 1.** Let, for instance,  $n_j = h_{j+1}/2$ , then standard rank one calculations (see [1],[4]) allow us to

$$\begin{aligned} \mu(T^{n_j} A \cap B) &\approx \\ &\approx \frac{1}{r_j} \sum_{i=1}^{r_j/2} \mu(T^{-d_0 i} A \cap B) + \frac{1}{r_{j+1}} \sum_{i=1}^{r_{j+1}} \mu(T^{-h_{j+1}/2} T^{-d_1 i} A \cap B \cap D_0(j)), \end{aligned} \quad (*)$$

where  $d_0 = 0$ ,  $d_1 = 1$ ,

$$D_0(j) = \bigsqcup_{i=0}^{h_{j+1}/2-1} E_{j+1}, \quad \mu(D_0) \approx \frac{1}{2}.$$

We rewrite (\*) in the form

$$T^{n_j} \approx_w \frac{1}{2}I + \hat{D}_0(j) \frac{1}{r_{j+1}} \sum_{i=1}^{r_{j+1}} T^{-h_{j+1}/2} T^{-i}.$$

From the ergodicity of  $T$  we get

$$\frac{1}{r_{j+1}} \sum_{i=1}^{r_{j+1}} T^{-h_{j+1}/2} T^{-i} \approx \Theta,$$

hence,

$$T^{n_j} \approx_w \frac{1}{2}I + \frac{1}{2}\Theta.$$

**Example 2.** Let  $n_j = h_{j+1}/2 + k_j$ ,  $0 \leq |k_j| \leq h_j$ ,  
we have

$$\begin{aligned} \mu(T^{n_j} A \cap B) &\approx \mu(T^{k_j} A \cap B \cap D(j)) + \frac{1}{r_j} \sum_{i=1}^{r_j} \mu(T^{k(1,j)} T^{-i} A \cap B \cap D_1(j)) + \\ &+ \frac{1}{r_{j+1}} \sum_{i=1}^{r_{j+1}} \mu(T^{-h_{j+1}+n_j} T^{-i} A \cap B \cap D_0(j)), \end{aligned}$$

where

$$D(j) = U_j \cap \bigsqcup_{i=|k_j|}^{h_j} T^i E_j, \quad D_1(j) = U_j \cap \bigsqcup_{i=0}^{|k_j|-1} T^i E_j, \quad U_j = \bigsqcup_{i=h_{j+1}/2}^{h_{j+1}} T^i E_{j+1}.$$

From the ergodicity of  $T$  we get

$$\begin{aligned} \mu(T^{n_j} A \cap B) &\approx \mu(T^{k_j} A \cap B \cap D(j)) + \left( \frac{h_j - |k_j|}{2h_j} + \frac{1}{2} \right) \mu(A) \mu(B), \\ T^{n_j} &\approx_w \hat{D}(j) T^{k_j} + \left( \frac{h_j - |k_j|}{2h_j} + \frac{1}{2} \right) \Theta. \end{aligned}$$

If  $n_j$  is not mixing, then from Lemma 2 it follows that  $k_j$  is not mixing too. We get the following alternative

*either*

$$T^{n_j} \approx_w \hat{D}(j) T^{k_j} + \frac{3}{4} \Theta, \quad \mu(D(j)) \approx \frac{1}{4}, \quad |k_j|/h_j \approx 1/2,$$

*or*

$$T^{n_j} \approx_w \hat{D}(j) T^{k_j} + \frac{1}{2} \Theta, \quad \mu(D(j)) \approx \frac{1}{2}, \quad k_j/h_j \approx 0.$$

**In fact** we have a strong approximation

$$T^{n_j} \approx \hat{D}(j) T^{k_j} + \hat{Y}(j) T^{n_j},$$

where  $Y(j) = X \setminus D(j)$  and

$$\hat{Y}(j) T^{n_j} \approx_w (1 - \mu(D(j))) \Theta.$$

**Let  $k_j$  be not bounded**, then there is  $k'_j$ ,  $k'_j \ll k_j$  such that

$$T^{k_j} \approx \hat{D}'(j) T^{k'_j} + \hat{Y}'(j) T^{n_j},$$

$$T^{n_j} \approx \hat{D}(j)(\hat{D}'(j)T^{k'_j} + \hat{Y}'(j)T^{k_j}) + \hat{Y}(j)T^{n_j}.$$

If  $k'_j$  is not bounded, then for  $k''_j \ll k'_j$  we have

$$T^{n_j} \approx \hat{D}(j) \left[ \hat{D}'(j) \left( \hat{D}''(j)T^{k''_j} + \hat{Y}''(j)T^{k'_j} \right) + \hat{Y}'(j)T^{k_j} \right] + \hat{Y}(j)T^{n_j}.$$

Omitting  $(j)$  in  $D(j), \dots Y''(j)$  we rewrite the latter by

$$T^{n_j} \approx \hat{D}\hat{D}'\hat{D}''T^{k''_j} + \hat{D}\hat{D}'Y''T^{k'_j} + \hat{D}\hat{Y}'T^{k_j} + \hat{Y}T^{n_j}.$$

And so on. However the number of iterations  $'$  must be finite. Indeed,  $D, D', D''$  are special pieces of different towers, so

$$\mu(D \cap D' \cap D'') \approx \mu(D)\mu(D')\mu(D'') \approx \frac{1}{2^m}, \quad 3 \leq m \leq 6$$

(they are almost independent), and

$$\hat{Y}T^{n_j} \approx_w \mu(Y)\Theta,$$

$$\hat{D}\hat{Y}'T^{k_j} \approx_w \mu(D)\mu(Y')\Theta$$

(this follows from  $\hat{Y}'T^{k_j} \approx_w \mu(Y')\Theta$  and  $\mu(D \cap Y') \approx \mu(D)\mu(Y')$ ),

$$\hat{D}\hat{D}'Y''T^{k'_j} \approx_w \mu(D)\mu(D')\mu(Y'')\Theta.$$

If all sequences  $k''_j \dots'$  are unbounded, then

$$\mu(D \cap D' \dots \cap D'' \dots) \approx 0,$$

$$T^{n_j} \rightarrow \Theta,$$

a contradiction ( $n_j$  is non-mixing). Thus, there is  $k''_j \dots' = k$ , and  $m$  such that

$$\mu(D \cap D' \cap \dots \cap D'' \dots') \rightarrow \frac{1}{2^m}, \quad \text{so,} \quad \hat{D}\hat{D}' \dots \hat{D}'' \dots' T^k \rightarrow_w \frac{1}{2^m} T^k,$$

$$T^{n_j} \rightarrow \frac{1}{2^m} T^k + \left(1 - \frac{1}{2^m}\right) \Theta.$$

### 3 Mixing sequences.

Now we prove Lemma 1. To a reader who is familiar with Adams' approach we can explain a proof in "two words". Adams found a total control of mixing for staircase transformation based on mixing sequences  $m_j \in [h_j, Ch_j]$ . For this he produced a non-trivial method to control mixing properties ( $P(j) \approx \Theta$ ) of averaging operators in a form

$$P(j) = \frac{1}{r(j)} \sum_{i=1}^{r(j)} T^{d(j)i},$$

where some special sequences  $r(j), d(j)$  ( $r(j) \rightarrow \infty$ ). The case  $C > d(m) > 0$  is trivial ( $T$  is totally ergodic); the case  $d(j) \rightarrow \infty$  is of interest. Adams found a special number  $q(m)$  such that  $q(j) \ll r(j)$  and for large  $L$  ( $L = L(j)$  tends to infinity very slowly)

$$A(j) = \frac{1}{L} \sum_{i=1}^L T^{q(j)d(j)i} \approx \Theta.$$

Then for all  $n$

$$T^n A(j) \approx \Theta$$

holds, so

$$P(j) \approx \Theta$$

as a convex sum of  $T^n A(j)$ . The principle personage  $q(j)$  has been extremely resourcefully selected by Adams: he found  $q(j), h_{p(j)} < q(j)d(j) < 2h_{p(j)}$ , such that

$$T^{h(j)d(j)}, T^{2h(j)d(j)}, \dots, T^{Lh(j)d(j)} \approx_w \Theta.$$

Hence,

$$\begin{aligned} A(j)^* A(j) &\approx_w \Theta, \\ A(j) &\approx \Theta, \quad P(j) \approx \Theta. \end{aligned}$$

What is changed in our double staircase situations? Almost nothing. Again we have a starting mixing sequence  $[h_j, Ch_j]$  ( $1 \ll C$ ). Again we can approximate  $T^{m_j}$  by similar averaging operators (now we deal with 5 approximating

operators instead of Adams' 3 operators, but all operators are of the same nature). All is similar except one thing: now the case

$$d(m) = 0$$

may appear. Here we have the following effect: the image  $T^m E_j$  of our the base  $E_j$  has a flat part that is situated in one of floors  $T^k E_j$ ,  $0 \leq k \leq h_j$ . For example, the case  $n_j = h_{j+1}/2$  (it has been considered above) gives

$$\begin{aligned} 2\mu(T^{n_j} A \cap B) &\approx \mu(A \cap B) + \frac{1}{r_{j+1}} \sum_{i=1}^{r_{j+1}} \mu(T^{-i} A \cap B) = \\ &= \frac{2}{r_j} \sum_{i=1}^{r_j/2} \mu(T^{-di} A \cap B) + \frac{1}{r_{j+1}} \sum_{i=1}^{r_{j+1}} \mu(T^{-h_{j+1}/2} T^{-d_0 i} A \cap B), \end{aligned}$$

where  $d = 0$ ,  $d_0 = 1$ . (A half of the image  $T^m E_j$  is in  $E_j$ .)

For  $n_j = h_{j+1} + h_j$  we now have  $d = 1$ ,  $d_0 = 1$ ,

$$T^{n_j} \approx_w \frac{1}{r_j} \sum_{i=1}^{r_j/2} T^{-di} + \frac{1}{2r_{j+1}} \sum_{i=1}^{r_{j+1}} T^{-h_{j+1}/2 - h_j} T^{-d_0 i} \approx_w \Theta. \quad (1)$$

**Generally** we get an approximation

$$T^{m_j} \approx_w \sum_{s=0}^4 \hat{D}_s(j) T^{k(s,j)} P_s(j),$$

where  $D_s(j)$  tile together the space  $X$ ,  $|k(s,j)| \leq h_j$ ,

$$P_s(j) = \frac{1}{r(s,j)} \sum_{i=1}^{r(s,j)} T^{d(s,j)i}.$$

Let for instant  $m_j/h_{j+1} \approx 0.2$ , then we have the following situation:

$$\begin{aligned} r(0,j) &= h_{j+1} - 1, \quad r(1,j) = r(2,j) \approx 0.3r_j, \quad r(3,j) = r(4,j) \approx 0.2r_j; \\ \mu(D_0(j)) &\approx 0.2, \quad \mu(D_1(j)) + \mu(D_2(j)) \approx 0.6, \quad \mu(D_3(j)) + \mu(D_4(j)) \approx 0.2; \\ d(0,j) &= 1, \quad d(1,j) = d(2,j) + 1 \approx 0.3r_j, \quad d(3,j) + 1 = d(4,j) \approx -0.1r_j. \end{aligned}$$

Sometimes certain  $D_s(j)$  could be vanishing. For example, in our formula (1) (as  $m_j = 0.5h_{j+1} + h_j$ ) we had  $\mu(D_0(j)) = 0.5$ ,  $\mu(D_4(j)) = 0.5$ ,  $d(0,j) =$

$d_0 = 1 = d = d(4, j)$  (here  $D_0(j)$  is a part of  $j$ -tower that is situated under the first stairs array of spacers,  $D_4(j)$  – under the second one). We will not weary the reader with tedious calculations, we may claim: for

$$m_j \in [h_j, h_{j+1}] \setminus [h_{j+1}/2 - h_j, h_{j+1}/2 + h_j]$$

the corresponding  $d(s, j)$  are non-zero. Thus, Adams' method guarantees our  $\{m_j\}$  to be mixing:

$$T^{m_j} \approx_w \sum_{s=0}^4 \hat{D}_s(j) T^{k(s,j)} P_s(j) \approx \Theta.$$

Lemma 1 is proved. Theorem 1 is proved.

#### 4 Remarks on related infinite transformations

1. There is a simple approach to construct non-mixing Gaussian automorphisms (see also [3], p. 92.) and Poisson suspensions with explicit countable WLP. Following [5] let's consider double Sidon rank one infinite transformation  $T$ : a rank one construction with a double spacer sequence

$$s_j(1), s_j(2), \dots, s_j(r'_j), s_j(1), s_j(2), \dots, s_j(r'_j)$$

satisfied the condition

$$h_j \ll s_j(1) \ll s_j(2) \ll \dots \ll s_j(r'_j - 1) \ll s_j(r'_j).$$

Easily controlling mixing sequences we get simply  $WLP(T) = \{0, 2^{-m}T^n\}$  (let us remark also that the centralizer of  $T$  is trivial), so

$$WLP(\mathbf{exp}(T)) = \{\Theta, \mathbf{exp}(2^{-m}T^n) : m = 1, 2, \dots, n \in \mathbf{Z}\}.$$

The work [2] gives also a natural way to construct a double spacer map  $S$  with  $WLP(S) = \{0, 2^{-m}S^n\}$ .

2. A calculation  $WLP(T)$  for double staircase transformations  $T$  of an infinite measure space is an interesting (and maybe hard) problem as  $r_j \sim h_j$ .

*Conjecture.* Any staircase transformation of an infinite measure space is mixing as  $r_j \rightarrow \infty$ .



The case  $r_j/h_j \rightarrow 0$  is solved ( the author presented a proof at Roscoff conference "Stochastic properties of dynamical systems and random walks", June, 2010).

**Problem.** *Prove the mixing for the " $r_j = h_j$ " staircase transformation.*  
The calculation of  $|P \cap P + m|$  for

$$P = \{p : p = dh + \sum_{i=0}^{d-1} i, d = 1, 2, \dots, [(1 - \varepsilon)h]\}$$

is naturally connected to the problem and seems non-trivial.

## References

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